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A MULTIGRID METHOD FOR STEADY INCOMPRESSIBLE NAVIER–STOKES EQUATIONS BASED ON PARTIAL FLUX SPLITTING

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SUMMARY

Flux splitting is applied to the convective part of the steady Navier–Stokes equations for incompressible flow. Partial upwind differences are introduced in the split first-order part, while central differences are used in the second-order part. The discrete set of equations obtained is positive, so that it can be solved by collective variants of relaxation methods. The partial upwinding is optimized in the same way as for a scalar convection–diffusion equation, but involving several Peclet numbers. It is shown that with the optimum partial upwinding accurate results can be obtained. A full multigrid method in W-cycle form, using red–black successive under-relaxation, injection and bilinear interpolation, is described. The efficiency of this method is demonstrated.

KEY WORDS Steady Navier–Stokes equations Partial flux splitting Multigrid methods

INTRODUCTION

The flux-vector-splitting method was introduced by Steger and Warming[1] to solve unsteady Euler equations. Further, it was shown by Jespersen[2] that the flux-vector-splitting method can also be used on the steady Euler equations to generate discrete equations which form a positive set so that a solution by relaxation methods, in multigrid form, is possible.

The concept of flux-vector splitting was extended to steady Navier–Stokes equations for incompressible flow by the author[3] and an optimized partial flux-splitting formulation was developed.[4] The set of equations obtained is positive and can be solved by relaxation methods.

In this paper, after a short review of the principles, a multigrid version of the method is presented and its efficiency is demonstrated.

FLUX-VECTOR SPLITTING FOR STEADY NAVIER–STOKES EQUATIONS

The steady Navier–Stokes equations for an incompressible fluid are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}=\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{2}$$

$$c^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0, \tag{3}$$

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where u and v are the Cartesian components of velocity, c is a reference velocity introduced to homogenize the eigenvalues of the system matrices, ν is kinematic viscosity and p is pressure divided by density.

In system form the set of equations (1)–(3) becomes

$$\begin{bmatrix} u & 0 & 1 \\ 0 & u & 0 \\ c^2 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \\ p \end{bmatrix} + \begin{bmatrix} v & 0 & 0 \\ 0 & v & 1 \\ 0 & c^2 & 0 \end{bmatrix} \frac{\partial}{\partial y} \begin{bmatrix} u \\ v \\ p \end{bmatrix} = \begin{bmatrix} \nu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 0 \end{bmatrix} \Delta \begin{bmatrix} u \\ v \\ p \end{bmatrix} \tag{4}$$

or symbolically

$$\mathbf{A} \frac{\partial \xi}{\partial \mathbf{x}} + \mathbf{B} \frac{\partial \xi}{\partial \mathbf{y}} = \mathbf{D} \left(\frac{\partial^2 \xi}{\partial \mathbf{x}^2} + \frac{\partial^2 \xi}{\partial \mathbf{y}^2} \right). \tag{5}$$

The matrices **A** and **B** in (5) have both positive and negative eigenvalues, while **D** has non-negative eigenvalues. Furthermore, **A** and **B** have a complete set of eigenvectors. As a consequence, the set of equations (5) forms a convective–diffusive set.

Obviously it is always possible to split the matrices **A** and **B** into non-negative and non-positive parts, so that a split form of the set (5) becomes

$$\mathbf{A}^+ \frac{\partial \xi}{\partial x} + \mathbf{A}^- \frac{\partial \xi}{\partial x} + \mathbf{B}^+ \frac{\partial \xi}{\partial y} + \mathbf{B}^- \frac{\partial \xi}{\partial y} = \mathbf{D} \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right), \tag{6}$$

where $\mathbf{A}^+$ and $\mathbf{B}^+$ have non-negative eigenvalues while $\mathbf{A}^-$ and $\mathbf{B}^-$ have non-positive eigenvalues.

The splitting of the matrices **A** and **B** is not unique. However, since the matrices have a complete set of eigenvectors, an obvious way of splitting is a splitting based on the eigenvalue matrices, as proposed by Steger and Warming.[1]

By denoting the eigenvalue matrices of **A** and **B** by Λ_A and Λ_B and the left eigenvector matrices by $\mathbf{X}_A$ and $\mathbf{X}_B$, obviously

$$\mathbf{A} = \mathbf{X}_A^{-1} \Lambda_A \mathbf{X}_A, \qquad \mathbf{B} = \mathbf{X}_B^{-1} \Lambda_B \mathbf{X}_B.$$

The eigenvalue matrices can be split into

$$\Lambda_A = \Lambda_A^+ + \Lambda_A^-, \qquad \Lambda_B = \Lambda_B^+ + \Lambda_B^-,$$

where $\Lambda_A^\pm$ and $\Lambda_B^\pm$ are diagonal matrices constructed by collecting the positive and negative entries from Λ_A and Λ_B.

Split matrices are then obtained by

$$\mathbf{A}^\pm = \mathbf{X}_A^{-1} \Lambda_A^\pm \mathbf{X}_A, \qquad \mathbf{B}^\pm = \mathbf{X}_B^{-1} \Lambda_B^\pm \mathbf{X}_B.$$

For the Navier Stokes equations (5), the eigenvalues of the system matrices **A** and **B** are

$$\lambda_{1A} = u, \qquad \lambda_{2A} = \frac{u + \sqrt{(u^2 + 4c^2)}}{2}, \qquad \lambda_{3A} = \frac{u - \sqrt{(u^2 + 4c^2)}}{2},$$

$$\lambda_{1B} = v, \qquad \lambda_{2B} = \frac{v + \sqrt{(v^2 + 4c^2)}}{2}, \qquad \lambda_{3B} = \frac{v - \sqrt{(v^2 + 4c^2)}}{2}.$$

Obviously λ_{2A} and λ_{2B} are always positive, λ_{3A} and λ_{3B} are always negative, while λ_{1A} and λ_{1B} change sign with u and v.

Hence

$$\Lambda_A^+ = \text{diag}(u^+, \lambda_{2A}, 0), \qquad \Lambda_A^- = \text{diag}(u^-, 0, \lambda_{3A}),$$
$$\Lambda_B^+ = \text{diag}(v^+, \lambda_{2B}, 0), \qquad \Lambda_B^- = \text{diag}(v^-, 0, \lambda_{3B}),$$

with

$$u^+ = \max(u, 0), \qquad u^- = \min(u, 0), \qquad v^+ = \max(v, 0), \qquad v^- = \min(v, 0).$$

According to the procedure of Steger and Warming, the split matrices become

$$\mathbf{A}^+ = \mathbf{X}_A^{-1}\Lambda_A^+\mathbf{X}_A = \begin{bmatrix} \hat{u}^+ & 0 & \alpha_1 \\ 0 & u^+ & 0 \\ \alpha_1 c^2 & 0 & a \end{bmatrix}, \qquad \mathbf{A}^- = \mathbf{X}_A^{-1}\Lambda_A^-\mathbf{X}_A = \begin{bmatrix} \hat{u}^- & 0 & \alpha_2 \\ 0 & u^- & 0 \\ \alpha_2 c^2 & 0 & -a \end{bmatrix},$$

$$\mathbf{B}^+ = \mathbf{X}_B^{-1}\Lambda_B^+\mathbf{X}_B = \begin{bmatrix} v^+ & 0 & 0 \\ 0 & \hat{v}^+ & \beta_1 \\ 0 & \beta_1 c^2 & b \end{bmatrix}, \qquad \mathbf{B}^- = \mathbf{X}_B^{-1}\Lambda_B^-\mathbf{X}_B = \begin{bmatrix} v^- & 0 & 0 \\ 0 & \hat{v}^- & \beta_2 \\ 0 & \beta_2 c^2 & -b \end{bmatrix},$$

where

$$\hat{u}^+ = \alpha_1 u + a, \qquad \hat{u}^- = \alpha_2 u - a, \qquad \hat{v}^+ = \beta_1 v + b, \qquad \hat{v}^- = \beta_2 v - b,$$

with

$$\alpha_1 = 0{\cdot}5(1+\alpha), \qquad \alpha_2 = 0{\cdot}5(1-\alpha), \qquad \beta_1 = 0{\cdot}5(1+\beta), \qquad \beta_2 = 0{\cdot}5(1-\beta),$$
$$a = c^2/\sqrt{(u^2+4c^2)}, \quad b = c^2/\sqrt{(v^2+4c^2)}, \quad \alpha = u/\sqrt{(u^2+4c^2)}, \quad \beta = v/\sqrt{(v^2+4c^2)}.$$

On a rectangular grid, using upwind differences in the first-order part (i.e. backward differences for terms with a plus sign and forward differences for terms with a minus sign) and central differences in the second-order part, the discretization of (6) is

$$\begin{aligned} \mathbf{C}\xi_{i,j} = {} & \mathbf{A}^+(1/\Delta x_w)\xi_{i-1,j} + (-\mathbf{A}^-)(1/\Delta x_e)\xi_{i+1,j} \\ & + \mathbf{B}^+(1/\Delta y_s)\xi_{i,j-1} + (-\mathbf{B}^-)(1/\Delta y_n)\xi_{i,j+1} \\ & + \mathbf{D}(1/\Delta x/\Delta x_w)\xi_{i-1,j} + \mathbf{D}(1/\Delta x/\Delta x_e)\xi_{i+1,j} \\ & + \mathbf{D}(1/\Delta y/\Delta y_s)\xi_{i,j-1} + \mathbf{D}(1/\Delta y/\Delta y_n)\xi_{i,j+1}, \end{aligned} \tag{7}$$

where C is the sum of the matrix coefficients on the right-hand side and where

$$\Delta x_w = x_{i,j} - x_{i-1,j}, \qquad \Delta x_e = x_{i+1,j} - x_{i,j}, \qquad \Delta x = 0{\cdot}5(\Delta x_w + \Delta x_e),$$
$$\Delta y_s = y_{i,j} - y_{i,j-1}, \qquad \Delta y_n = y_{i,j+1} - y_{i,j}, \qquad \Delta y = 0{\cdot}5(\Delta y_s + \Delta y_n).$$

It can be verified that for the Steger and Warming splitting the eigenvalues of the **C**-matrix are always positive.[3] As a consequence, the set of equations (7) is a vector-positive set. It can be solved by a vector variant of any scalar relaxation scheme.

PARTIAL FLUX-SPLITTING FORMULATION

Using full upwind differences, the set of equations (7) is positive, even without the diffusive part. Since the diffusive part contributes to the positivity for the momentum equations, a partial upwind formulation is possible for these equations, retaining the positivity.

For example, the momentum-x equation can be discretized as

$$\begin{aligned}\hat{u}^+[\theta_{xx}\delta_x^+ u+(1-\theta_{xx})\delta_x u]+\hat{u}^-[\theta_{xx}\delta_x^- u+(1-\theta_{xx})\delta_x u]\\ +v^+[\theta_{xy}\delta_y^+ u+(1-\theta_{xy})\delta_y u]+v^-[\theta_{xy}\delta_y^- u+(1-\theta_{xy})\delta_y u]\\ +\alpha_1\delta_x^+ p+\alpha_2\delta_x^- p=\nu(\delta_x^2 u+\delta_y^2 u),\end{aligned}$$

where

$$\begin{aligned}&\delta_x^+ u=(u_{i,j}-u_{i-1,j})/\Delta x_w, && \delta_x^- u=(u_{i+1,j}-u_{i,j})/\Delta x_e,\\ &\delta_x u=(\Delta x_e\delta_x^+ u+\Delta x_w\delta_x^- u)/2\Delta x,\\ &\delta_y^+ u=(u_{i,j}-u_{i,j-1})/\Delta y_s, && \delta_y^- u=(u_{i,j+1}-u_{i,j})/\Delta y_n,\\ &\delta_y u=(\Delta y_n\delta_y^+ u+\Delta y_s\delta_y^- u)/2\Delta y,\\ &\delta_x^2 u=(\delta_x^+ u-\delta_x^- u)/\Delta x, && \delta_y^2 u=(\delta_y^+ u-\delta_y^- u)/\Delta y.\end{aligned}$$

A similar discretization can be used on the momentum-y equation, involving θ_{yx} and θ_{yy}. The mass equation is to be discretized in a full upwind way.

The optimum values for the partial upwind coefficients θ_{xx}, θ_{xy}, θ_{yx} and θ_{yy} can be determined by expressing that the linearized form of the discrete equations -i.e. coefficients like $\hat{u}^+$, $\hat{u}^-$, v^+, v^-, . . . considered as being constant- -is of the form

$$e^{ux/\nu}e^{vy/\nu}.$$

In this way the optimum value of θ_{xx} is found to be given by

$$\theta_{xx}=\frac{u(\Delta x_w\sigma_e+\Delta x_e\sigma_w)/(\sigma_e-\sigma_w)-2\nu}{\hat{u}^+\Delta x_w-\hat{u}^-\Delta x_e}, \tag{8}$$

where

$$\sigma_w=\frac{1-e^{-u\Delta x_w/\nu}}{\Delta x_w}, \qquad \sigma_e=\frac{e^{u\Delta x_e/\nu}-1}{\Delta x_e}.$$

Similar expressions are found for θ_{xy}, θ_{yx} and θ_{yy}.

As is common practice for scalar equations, the expressions for the optimum partial upwind coefficients can be replaced by their expansions for small values of velocity, with a maximum of 1. Expansion of (8) leads to

$$\theta_{xx}=\min\{(Pe_{xx}/6),1\}, \tag{9}$$

where the Peclet number is

$$Pe_{xx}=(\hat{u}^+\Delta x_w-\hat{u}^-\Delta x_e)/\nu. \tag{10}$$

The results for the other coefficients are

$$\begin{aligned}&\theta_{xy}=\min\{(Pe_{xy}/6),1\}, && Pe_{xy}=(v^+\Delta y_s-v^-\Delta y_n)/\nu,\\ &\theta_{yx}=\min\{(Pe_{yx}/6),1\}, && Pe_{yx}=(u^+\Delta x_w-u^-\Delta x_e)/\nu,\\ &\theta_{yy}=\min\{(Pe_{yy}/6),1\}, && Pe_{yy}=(\hat{v}^+\Delta y_s-\hat{v}^-\Delta y_n)/\nu.\end{aligned} \tag{11}$$

NUMERICAL EXAMPLE

Figure 1 shows a well known GAMM backward-facing step problem,[5] discretized with a coarse grid with 42 elements. This grid is the coarsest of a series of four, the finest grid having 2688

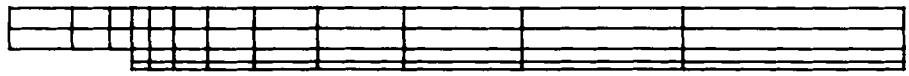

Figure 1. Backward-facing step problem discretized with a coarse grid

elements. In the construction of finer grids, the same stretching law is applied as used in the coarse grid.

In earlier versions[3,4] boundary equations were derived from the field equations by combining them in order to eliminate outgoing derivatives. These so-called consistent boundary equations were supplemented with obvious essential boundary conditions.

In this paper the boundary equations are constructed in the more classical way by taking inward discretizations of the normal momentum equation. It was found that the solutions obtained by both methods are nearly identical.

The boundary conditions and boundary equations are as follows.

At inlet

(1) $u = u_0(y)$ in which $u_0(y)$ is a parabolic profile with a mean velocity c
(2) $v = 0$
(3) p from the momentum-x equation in which an assumption of fully developed flow is used:

$$\delta_x^- p = \nu \delta_y^2 u.$$

At outlet

(1) $p = 0$
(2) $v = 0$
(3) u from an assumption of fully developed flow:

$$\delta_x^+ u = 0.$$

At solid boundaries

(1) $u = 0$
(2) $v = 0$
(3) p from an inward discretization of the normal momentum equation.

At horizontal parts of the boundary this is

$$\delta_y^{\pm} p = \nu \delta_y^{2\pm} v. \tag{12}$$

In (12) the inward discretization of the second derivative is obtained by Taylor expansion, using $\delta_y^{\pm} v = 0$ from the mass equation. For example, for a horizontal bottom boundary this results in

$$p_{i,j}/\Delta y_n = p_{i,j+1}/\Delta y_n - 2\nu v_{i,j+1}/\Delta y_n^2. \tag{13}$$

At the vertical part of the boundary the p-equation is

$$\delta_x^- p = \nu \delta_x^{2-} u$$

or

$$p_{i,j}/\Delta x_e = p_{i+1,j}/\Delta x_e - 2\nu u_{i+1,j}/\Delta x_e. \tag{14}$$

At the corner points in the bottom part of the boundary the mean value from equations (13) and (14) is taken.

Figure 2 shows the solution obtained with a successive under-relaxation method (relaxation factor 0·95) in red–black ordering for

$$Re = U_{max} h/\nu = 150,$$

where U_{max} is the maximum value of the velocity at the inlet section and h is the step height. The streamlines shown in Figure 2 were obtained by integration of the calculated velocity profiles. The reattachment length to step height ratio is about 6. This result is in accordance with the experimental value.[5]

Figure 3 shows the isobars, normalized according to

$$\bar{p} = Re(p - p_c)/\tfrac{1}{2}\rho U_{max}^2,$$

where p_c is the corner pressure.

MULTIGRID FORMULATION

A description of the basics of the multigrid method is not given here. The reader not familiar with the terminology used in this section is referred to the overview of Stüben and Trottenberg.[6]

All equations are normalized by bringing the coefficient of u, v and p in the central node, for the momentum-x, momentum-y and pressure (mass) equation respectively, to the value 1. As a result, field equations and boundary equations take a similar form. This allows the use of full weighting as restriction operator for defects of boundary equations and field equations.

Successive under-relaxation in red black form was chosen as relaxation algorithm. For a system of first-order equations the maximum relaxation factor for stability is 1 (not 2). Maximum convergence rate for a single-grid calculation was found to be obtained for a relaxation factor of 0·95. Although it is well known that red–black relaxation does not have optimum smoothing properties, this algorithm was chosen for its ease in vectorizing the code.

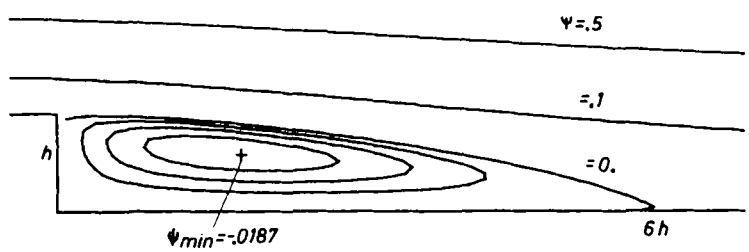

Figure 2. Streamline pattern for the backward-facing step problem, obtained at the finest grid

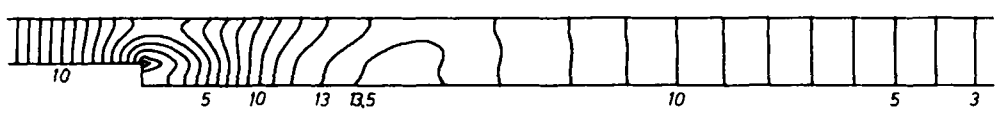

Figure 3. Pressure level pattern for the backward-facing step problem

A full approximation scheme was used. For the restriction operator, experiments were done with full weighting on both function values and defects, with injection for function values and full weighting for defects, and with injection for both. In the full weighting versions, experiments were done with full weighting at the boundaries and with weighting restricted to boundary points.

As a result of the normalization process, the sum of the weight factors in the full weighting for defects is 2. As is well known, with red–black ordering, the weight factor using injection for defects is not 2 but 1, i.e. half-injection is to be used. Bilinear interpolation was used as prolongation operator.

The classical cycle configurations were tried: V-cycle, F-cycle and W-cycle. It was found that the W-cycle performs best. Also the full multigrid method, i.e. using a nested iteration as starting cycle, was used. The cycle geometry is shown in Figure 4. Each dot represents a relaxation operation. The nested iteration also has a W-form.

The best efficiency of the multigrid cycle was found to be reached for the same relaxation factor as for single-grid calculations: $\omega = 0{\cdot}95$.

It was found that the performance is insensitive to the choice of the restriction operator. Using full weighting for defects is slightly more efficient than using injection, in terms of the required number of cycles. However, since injection requires less residue evaluations, in terms of work units the performance is about the same. The performance is also not sensitive to the precise weighting formula: algebraic weighting (i.e. weighting factors $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$) or geometric weighting (i.e. weighting factors taking into account the distances between nodes). Therefore, owing to its simplicity, injection for both functions and defects was retained for further use.

Figure 5 shows the convergence history for a single-grid calculation and a multigrid calculation. The initial condition is a flow with $v = 0$ and $p = 0$ everywhere and with u equal to the inlet profile in the upper part of the flow field and $u = 0$ in the lower part of the flow field. In the evaluation of the work of a cycle, on the finest grid, a relaxation and a residue calculation with the associated grid transfer are counted as one work unit.

The work done in each node of the cycle is indicated in Figure 4. In going down in the cycle, the 3 represents a residue evaluation in the coarse grid points of the next finer grid and its injection, a residue evaluation on the coarse grid to form the right-hand side in the coarse grid equations, and one relaxation. The work spent in a cycle is 4·125 work units. The work spent in the nested iteration is 2·141 work units.

CONCLUSIONS

It was shown that the flux-vector-splitting technique can be applied to steady Navier–Stokes equations in incompressible flow, leading to discrete equations that can be solved by vector

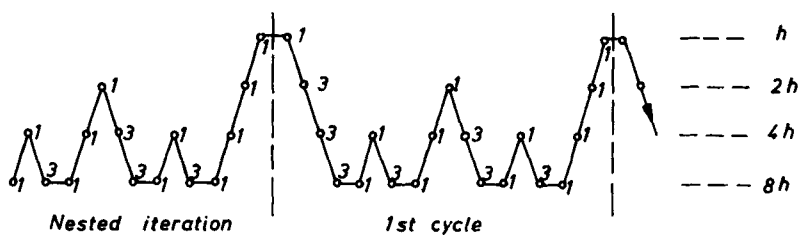

Figure 4. Geometry of the multigrid cycle

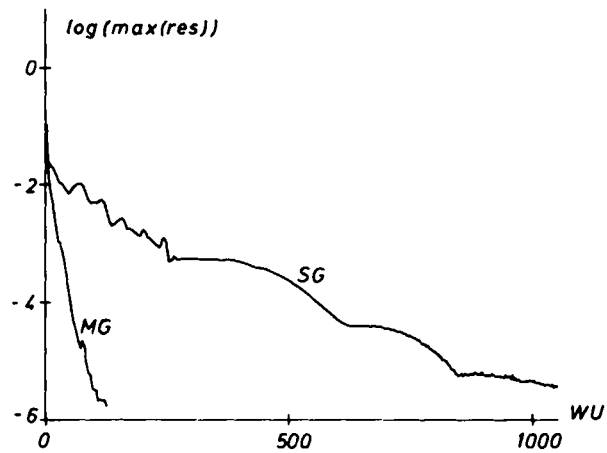

Figure 5. Convergence history for single-grid and multigrid red–black relaxation

variants of relaxation schemes. As a result of the partial upwinding, an accurate solution is obtained. By the use of the multigrid formulation a very efficient solution technique is realized.

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